assumption that

$$f(0) \geq \prod_{j=1}^{n-1}\left(1-\prod_{i=1}^{m} y_{ij}\right)+\prod_{i=1}^{m}\left(1-\prod_{j=1}^{n-1}\left(1-y_{ij}\right)\right) \geq 1, \qquad (4)$$

and from (3), we obtain by the induction assumption that

$$f(1) \geq \prod_{j=1}^{n}\left(1-\prod_{i=1}^{m-1} y_{ij}\right)+\prod_{i=1}^{m-1}\left(1-\prod_{j=1}^{n}\left(1-y_{ij}\right)\right) \geq 1. \qquad (5)$$

Since $f(y_{mn})$ is a polynomial in y_{mn} with degree 0 or 1, so from (4) and (5), we see that $f(y_{mn}) \geq 1$, and (1) holds also for $m+n=k+1$. Hence (1) holds in general and the inequality of the problem follows by the substitution $y=\dfrac{\sqrt{x_{ij}}}{1+\sqrt{x_{ij}}}$.

Also solved by Bruno Salgueiro Fanego, Viveiro, Spain; Ethan Gegner (student), Taylor University, Upland, IN; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5335:** *Proposed by Arkady Alt, San Jose, CA*

Prove that for any real $p>1$ and $x>1$ that

$$\frac{\ln x}{\ln(x+p)} \leq\left(\frac{\ln(x+p-1)}{\ln(x+p)}\right)^{p}.$$

Solution 1 by Ethan Gegner (student), Taylor University, Upland, IN

The weighted AM-GM inequality, followed by Jensen's inequality applied to the concave function $\ln x$ yields

$$\begin{aligned}(\ln x)^{1/p}(\ln(x+p))^{\frac{p-1}{p}} &\leq \frac{1}{p}\ln x+\frac{p-1}{p}\ln(x+p)\\ &\leq \ln\left(\frac{1}{p}x+\frac{p-1}{p}(x+p)\right)\\ &= \ln(x+p-1).\end{aligned}$$

Exponentiation by p and then rearranging yields the desired result.

Solution 2 by Bruno Salgueiro Fanego, Viveiro, Spain

The inequality is true for any real $p \geq 1$ and $x>1$, because

$$\left(\frac{\ln(x+p-1)}{\ln(x+p)}\right)^{p}-\frac{\ln x}{\ln(x+p)} \geq 1+p\left(\frac{\ln(x+p-1)}{\ln(x+p)}-1\right)-\frac{\ln x}{\ln(x+p)}$$

$$= \frac{p\ln\left(\dfrac{x+p-1}{x+p}\right) - \ln\left(\dfrac{x}{x+p}\right)}{\ln(x+p)}$$

$$= \frac{qy\ln(1-y^{-1}) - \ln(1-q)}{\ln y}$$

$$= \frac{q}{\ln y}\left(-\sum_{k=1}^{\infty} k^{-1}y^{1-k} + \sum_{k=1}^{\infty} k^{-1}q^{k-1}\right)$$

$$= \frac{q}{\ln y}\sum_{k=1}^{\infty} k^{-1}\left(q^{k-1} - y^{1-k}\right) \geq 0,$$

where we have used Bernoulli's inequality

$$(1+t)^p \geq 1 + pt \text{ for } t = \frac{\ln(x+p-1)}{\ln(x+p)} - 1 \geq -1.$$

Note that $p \geq 1, x > 1 \Rightarrow x+p-1 > 1, x+p > 1 \Rightarrow \ln(x+p-1), \ln(x+p) > 0$, the notation $y = x+p$ and $q = \dfrac{p}{y}$, the series expansion $\ln(1-u) = -\sum_{k=1}^{\infty} k^{-1}u^k$ for $u = y^{-1}$ and $u = q$ (observe that $0 < y^{-1}, q < 1$) and the fact that $q \geq y^{-1}$ with equality iff $p = 1 \Rightarrow q^{k-1} \geq \left(y^{-1}\right)^{k-1}$ for any integer $k \geq 1$.

Moreover, equality is attained iff it occurs in Bernouilli's inequality and in the inequality $q \geq y^{-1}$. Since there is equality in this last inequality iff $p = 1$ and in this case also in Bernoulli's inequality, we conclude that equality occurs iff $p = 1$.

Solution 3 by Paul M. Harms, North Newton, KS

All logarithms involved with the inequality are positive. Then the inequality is correct if the logarithm of the left side is less than the logarithm of the right side. Taking the natural logarithm of both sides and dividing by p the problem inequality is equivalent t o

$$\frac{\ln\ln x - \ln\ln(x+p)}{p} \leq \frac{\ln\ln(x+p-1) - \ln\ln(x+p)}{1},$$

Let $f(x) = \ln\ln x$ where $x > 1$. Multiplying both sides of the inequality by (-1) we can write the resulting inequality as

$$\frac{f(x+p) - f(x)}{(x+p) - x} \geq \frac{f(x+p) - f(x+p-1)}{(x+p) - (x+p-1)},$$

forms often associated with the Mean Value Theorem for derivatives.

Let the following letters and points be associated with each other:

$$A\left(x, f(x)\right),\ B\left((x+p), f(x+p)\right),\ C\left((x+p), f(x)\right),$$

$$E\left((x+p-1), f(x+p-1)\right),\ F\left((x+p), f(x+p-1)\right),$$

and let D be intersection of the line segment between A and B with the line segment between E and F.

Consider the right triangle $\triangle BEF$ and the similar right triangles $\triangle ABC$ and $\triangle DBF$. The left side of the last inequality is the ratio of the distances $\dfrac{BC}{AC} = \dfrac{BF}{DF}$ and the right side equals the ratio $\dfrac{BF}{EF}$.

Since $f^{'}(x) = \dfrac{1}{x \ln x} > 0$, and $f^{''}(x) = \dfrac{-1\,(1+\ln x)}{(x\ln x)^2} < 0$ for $x > 1$, the line segment from A to B is below the graph of $y = f(x)$. Point D then satisfies the distance inequality $DF < EF$ so we have $\dfrac{BF}{DF} \geq \dfrac{BF}{EF}$. The problem inequality is correct.

Solution 4 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

The inequality in the statement of the problem is equivalent to

$$\frac{\ln x}{\ln(x+p)} \leq \left(\frac{\ln(x+p-1)}{\ln(x+p)}\right)^p \iff \ln\left(\ln\left(x+p\right)\right)^{p-1} \leq \left(\ln\left(x+p-1\right)\right)^p. \qquad (*)$$

Knowing that $\ln x > 0$ and using the AM-GM inequality, we have:

$$\ln x\left(\ln\left(x+p\right)\right)^{p-1} \leq \left(\frac{\ln x + (p-1)\ln(x+p)}{p}\right)^p = \left(\ln \sqrt[p]{x(x+p)^{p-1}}\right)^p \leq (\ln(x+p-1))^p$$

for every $p > 1$ and $x > 1$. Using the fact that $\ln x$ is an increasing function, we deduce that $(*)$ is true and also the equivalent inequality in the statement of the problem.

Also solved by Ed Gray, Highland Beach, FL; Kee-Wai Lau, Hong Kong, China; Haroun Meghaichi (student, University of Science and Technology, Houari Boumediene), Algiers, Algeria; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Albert Stadler, Herrliberg, Switzerland, and the proposer.

- **5336:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Caculate:

$$\sum_{k=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{k}-\ln\left(k+\frac{1}{2}\right)-\gamma\right).$$

Solution 1 by Perfetti Paolo, Department of Mathematics, Tor Vergata University, Rome, Italy

The first item we employ is

$$\sum_{k=1}^{n}\frac{1}{k} = \ln n + \gamma_n,\ \ \gamma_n = \gamma + o(1), \qquad n(\gamma_n - \gamma) \to 1/2.$$